

# Fuzzy Categories

## Work in Progress!

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### Abstract

Since categories are graphs with additional “structure”, one should start from fuzzy graphs in order to define a theory of fuzzy categories. Thus it makes sense to introduce categories whose morphisms are associated with a plausibility degree that determines to what extent it is possible to “go” from one object to another one. These categories are called *fuzzy categories*. Of course, the basic properties of these categories are similar but not identical to their ordinary counterparts. Thus, it is necessary to introduce notion like fuzzy commutative diagrams, fuzzy initial and fuzzy terminal objects, etc.

## 1 Introduction

Categories, which were invented by Samuel Eilenberg and Saunders Mac Lane, form a very high-level abstract mathematical theory that unifies all branches of mathematics. The standard reference on category theory is Mac Lane’s book-length introduction to categories [8]. Category theory plays a central role in modern mathematics and theoretical computer science, and, in addition, it is used in mathematical physics (e.g., see [3]), in software engineering [2], etc. However, what makes category theory even more interesting is that it is an alternative to set theory as a foundation for mathematics. Indeed, as Mac Lane [7] pointed out

It is now possible to develop almost all of ordinary Mathematics in a well-pointed topos [i.e., an ordinary category that has some additional properties] with choice and a natural number object. The development would seem unfamiliar; it has nowhere been carried out yet in great detail. However, this possibility does demonstrate one point of philosophical interest: The foundation of Mathematics on the basis of set theory (ZFC) is by no means the only possible one!

Roughly, a category is a universe that includes all mathematical objects of a particular form together with maps between them. These maps must obey a few basic principles. For example, the collection of all sets and functions between them forms the category that is traditionally denoted by **Set**. Obviously, one can form categories of fuzzy structures. For instance, Carol Walker [14] and Siegfried Gottwald [4] have defined such categories. However, some other categories of fuzzy structures emerged from efforts to define fuzzy models of linear logic. In particular, Michael Barr [1], Basil Papadopoulos and this author [10, 11, 13] have introduced such categories.

Not so surprisingly, a category of fuzzy structures is not a fuzzy structure itself. Indeed, Alexander Šostak [12] was the first researcher who had realized this. In order to remedy this situation, Šostak introduced a new structure that mimics the way fuzzy sets are defined as is evident from the following definition:

**Definition 1.1** An  $L$ -fuzzy category (where  $L$  is a GL-monoid) is a quintuple  $\mathcal{C} = (\text{Ob}(\mathcal{C}), \omega, M(\mathcal{C}), \mu, \circ)$  where  $\mathcal{C}_\perp = (\text{Ob}(\mathcal{C}), M(\mathcal{C}), \circ)$  is a usual (classical) category called the bottom frame of the fuzzy category  $\mathcal{C}$ ;  $\omega : \text{Ob}(\mathcal{C}) \rightarrow L$  is an  $L$ -subclass of the class of objects  $\text{Ob}(\mathcal{C})$  of  $\mathcal{C}_\perp$  and  $\mu : M(\mathcal{C}) \rightarrow L$  is an  $L$ -subclass of the class of morphisms  $M(\mathcal{C})$  of  $\mathcal{C}_\perp$ . Besides  $\omega$  and  $\mu$  must satisfy the following conditions:

- i) if  $f : X \rightarrow Y$ , then  $\mu(f) \leq \omega(X) \wedge (Y)$ ;
- ii)  $\mu(g \circ f) \geq \mu(g) * \mu(f)$  whenever composition  $g \circ f$  is defined, where  $*$  is a binary operator that obeys a number of rules;
- iii) if  $e_X : X \rightarrow X$  is the identity morphism, then  $\mu(e_X) = \omega(X)$ .

For reasons that will become clear later on, Šostak's approach is not the ideal solution to the problem of the “fuzzification” of category theory. Instead, by following a different path one should be to define an alternative form of fuzzy categories. In particular, since any category can be identified with a graph (the inverse is not true), I am using the results of fuzzy graph theory to define, what I think are, real fuzzy categories. This approach is justified by the fact that (meta)categories are introduced using notions from (meta)graph theory. Interestingly, Gérard Huet and Amokrane Saïbi [5] have followed a similar line of thought in order to define constructive categories and categorical structures.

One may wonder why someone should get into the trouble to define fuzzy categories. Indeed, this is a quite reasonable question since there is no room for more meaningless generalizations. However, if fuzziness, in particular, and *vagueness*, in general, are fundamental properties of this cosmos, then we should use this fact to define deviant Mathematics.<sup>1</sup> Thus, if this is true, then we could use fuzzy categories as a tool to develop the foundations for this deviant Mathematics.

## 2 Basic Ideas and Concepts

As was explained in the introduction, I will define fuzzy categories using known notions from fuzzy graph theory. Thus, it is important to recall the notions that are necessary to define fuzzy categories. Let us start with fuzzy graphs:

**Definition 2.1** A *fuzzy graph* [9] consists of a collection of nodes and a collection of arrows between these nodes. Each arrow must have a specific *domain* node (i.e., its source), a *codomain* node (i.e., its target), and a plausibility degree, which expresses the grade to which it is possible to go from the domain to the codomain of an arrow.

Obviously, it is possible to have two or more different arrows that have the same domain and codomain, but, clearly, different plausibility degrees. The notation “ $f : A \xrightarrow[\rho]{} B$ ” means that  $f$  is an arrow that goes from  $A$  (the domain) to  $B$  (the codomain) with plausibility degree that is equal to  $\rho \in [0, 1]$ . Alternatively, one can use the following notation:

$$A \xrightarrow[\rho]{} B.$$

**Definition 2.2** Let  $k > 0$ . In a fuzzy graph  $\mathcal{G}$  a path from a node  $X$  to a node  $\Psi$  of length  $k$  is a sequence  $(f_1, f_2, \dots, f_k)$  of arrows, which are not necessarily distinct, such that

$$X \xrightarrow[\rho_k]{} A_{k-1} \xrightarrow[\rho_{k-1}]{} \dots A_1 \xrightarrow[\rho_1]{} \Psi$$

and the plausibility degree of the path is  $\min_{i=1}^k \rho_i$ .

**Remark 1** We can use any t-conorm, but for reasons of simplicity we use min.

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<sup>1</sup>Obviously, according to quantum mechanics our world is vague, but probabilistic systems do not fundamentally differ from their deterministic counterparts (e.g., nondeterministic and deterministic Turing machines have exactly the same computational power). Nevertheless, it seems that this does not apply to fuzziness.

Having defined fuzzy arrows and fuzzy arrow composition, we can proceed with the definition of fuzzy categories.

**Definition 2.3** A fuzzy category  $\mathcal{C}$  comprises

- i) a collection of entities called *objects*;
- ii) a collection of entities called *arrows* or *morphisms*;
- iii) operations assigning to each  $\mathcal{C}$ -arrow  $f$  a  $\mathcal{C}$ -object  $A = \text{dom } f$ , its domain, a  $\mathcal{C}$ -object  $B = \text{cod } f$ , its codomain, and a plausibility degree  $\rho = \text{p } f$ . Typically, the plausibility degree is a real number belonging to the unit interval. These operations on  $f$  are indicated by displaying  $f$  as an arrow starting from  $A$  and ending at  $B$  with plausibility degree  $\rho$ :

$$A \xrightarrow[\rho]{f} B \quad \text{or} \quad f : A \xrightarrow[\rho]{} B;$$

- iv) an operation assigning to each pair  $(g, f)$  of arrows with  $\text{dom } g = \text{cod } f$ , an arrow  $g \circ f$ , the *composite* of  $f$  and  $g$ , having  $\text{dom}(g \circ f) = \text{dom } f$ ,  $\text{cod}(g \circ f) = \text{cod } g$ , and  $\text{p}(g \circ f) = \min\{\text{p } f, \text{p } g\}$ . This operation and the previous three are subject to the *associative law*. Given the configuration

$$A \xrightarrow[\rho_1]{f} B \xrightarrow[\rho_2]{g} C \xrightarrow[\rho_3]{h} D$$

of  $\mathcal{C}$ -objects and  $\mathcal{C}$ -arrows, then  $h \circ (g \circ f) = (h \circ g) \circ f$ ;

- v) an *assignment* to each  $\mathcal{C}$ -object  $B$  of a  $\mathcal{C}$ -arrow  $\mathbf{1}_B : B \xrightarrow{1} B$ , called the *identity arrow on  $B$* , such that the following *identity law* holds true:

$$\mathbf{1}_B \circ f = f \quad \text{and} \quad g \circ \mathbf{1}_B = g$$

for any  $\mathcal{C}$ -arrows  $f : A \xrightarrow[\rho_f]{} B$  and  $g : B \xrightarrow[\rho_g]{} A$ .

**Remark 2** Obviously, every ordinary category is a fuzzy category with arrows that have plausibility degree equal to 1.

**Example 2.1** Let us give a relatively simple example of a fuzzy category that has as objects sets. Assume that  $f : X \rightarrow Y$  is function. Then we say that  $f$  is an arrow from  $X$  to  $Y$  with plausibility degree  $\lambda$  if there are fuzzy subsets that are characterized by the functions  $A : X \rightarrow \mathbf{I}$  and  $B : Y \rightarrow \mathbf{I}$  such that  $B(f(x)) - A(x) \geq \lambda$ , for all  $x \in X$ . Assume that  $f : X \xrightarrow{\lambda_1} Y$  and  $g : Y \xrightarrow{\lambda_2} Z$  are two arrows. Then since  $B(f(x)) - A(x) \geq \lambda_1$  and  $C(g(f(x))) - B(f(x)) \geq \lambda_2$ , for all  $x \in X$ , one concludes that  $C(g(f(x))) - A(x) \geq \lambda_3$ , for all  $x \in X$  and where  $\lambda_3 = \min\{\lambda_1, \lambda_2\}$ . Thus, the composite arrow  $g \circ f : X \xrightarrow{\lambda_3} Z$  exists and can be defined from its constituents. It is not difficult to see that the associative law holds also. Clearly, the identity arrow for some object  $X$  is the identity function of this set. Also, it is almost trivial to see that the identity law holds. The resulting fuzzy category will be called **FSet**.

**Remark 3** The objects of a fuzzy category can be fuzzy structures, but this is something that should not concerns us. After all, category theory is about arrows and their properties and not about objects.

**Example 2.2** Assume that  $R : X \times X \rightarrow [0, 1]$  is a fuzzy relation such that  $R(x, x) = 1$  for all  $x \in X$  and  $R(x, y) * R(y, z) \leq R(x, z)$  for all  $x, y, z \in X$  and where  $*$  is a t-norm. A fuzzy relation with these properties is called a *\*-fuzzy preorder*. When the t-norm is function min, then it will be called just fuzzy preorder. Any fuzzy preorder relation  $R$  determines a fuzzy preorder category  $P$  in which the arrows  $p \xrightarrow{\rho} p'$  are exactly those pairs  $\langle p, p' \rangle$  for which  $R(p, p') = \rho$ . The fuzzy relation is transitive, which implies that there is a unique way of composing arrows. Also, the fuzzy relation is reflexive and so there are the necessary identity arrows.

**Example 2.3** Let  $\wedge$  be the only object of a category and let us identify all arrows of this category with their plausibility degrees. For example  $\wedge \xrightarrow{\lambda} \wedge$  is the arrow  $\lambda$  whose plausibility degree is obviously  $\lambda$ . Given two arrows  $\lambda_1$  and  $\lambda_2$ , and assuming that  $\wedge$  denotes the minimum, as is usually the case, then  $\lambda_1 \circ \lambda_2 = \lambda_1 \wedge \lambda_2$ . Assume there is an arrow  $1_\wedge$  such that  $1_\wedge \circ \lambda = \lambda$  and  $\lambda \circ 1_\wedge = \lambda$  for all arrows  $\lambda$ , then according to the definition of arrow composition this arrow is the identity arrow, that is,  $1_\wedge = 1$ . Note that it is quite possible to have more than one arrow that has plausibility degree equal to one, nevertheless, for our purposes these arrows will behave exactly like 1 does. In different words, they will be isomorphic, but we will say more about isomorphisms in a while.

**Example 2.4** A category is a *deductive system* (for example, see [6] for a thorough description of this categories-as-deductive-systems paradigm). In this paradigm, objects are seen as *formulas*, arrows as *proofs* (or deductions), and an operation on arrows as a *rule of inference*. In particular, each arrow  $f : A \rightarrow B$  is thought of as the “reason” why  $A$  entails  $B$ . Thus, the identity law is the reason why  $A$  entails  $A$ , for all  $A$  objects (formulas) and the associative law becomes the following rule of inference:

$$\frac{f : A \longrightarrow B \quad g : B \longrightarrow C}{f \circ g : A \longrightarrow C}$$

Similarly, a fuzzy category is a *fuzzy deductive system* in which objects may be *fuzzy formulas* (remember, the objects of any fuzzy category are not necessarily “crisp”), arrows are *fuzzy deductions*, and the associative law is the following fuzzy inference:

$$\frac{f : A \xrightarrow{\rho_f} B \quad g : B \xrightarrow{\rho_g} C}{f \circ g : A \xrightarrow{\rho_{f \circ g}} C}$$

**Commutative diagrams** In category theory *commutative diagrams* play the role equations play in algebra. In the simplest case a commutative diagram can be identified with two different paths starting from the same object  $A$  and ending with the same object  $B$  in which the composition of the arrows that make up the first path and the composition of the arrows of the second path yield two arrows that have the same effect (i.e., when applied to the same object(s), they yield the same result). In general, when dealing with fuzzy arrows we need to stick to this requirement, but we distinguish at least two different cases. In the first case, the plausibilities must be exactly the same while in the second case, they must be greater than a specific minimum. In particular, assume that

$$\begin{array}{c} A \xrightarrow[\lambda_n]{f_n} \dots \xrightarrow[\lambda_1]{f_1} B \\ A \xrightarrow[\rho_m]{g_m} \dots \xrightarrow[\rho_1]{g_1} B \end{array}$$

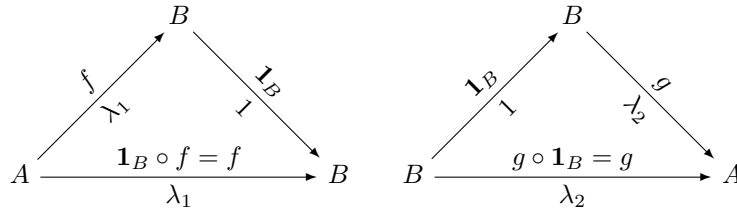
are two paths. Then these paths form a *strong* commutative diagram provided that

$$\min\{\lambda_1, \dots, \lambda_n\} = \min\{\rho_1, \dots, \rho_m\}.$$

Otherwise, we say that the two paths commute with plausibility degree  $\nu$ , where

$$\nu = \min\{\min\{\lambda_1, \dots, \lambda_n\}, \min\{\rho_1, \dots, \rho_m\}\}.$$

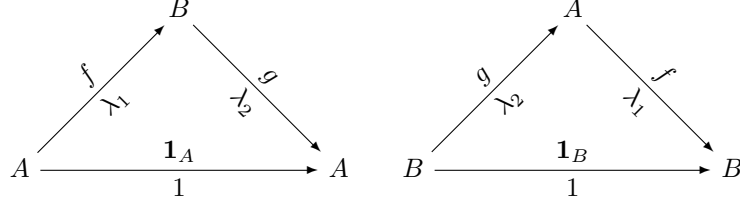
**Example 2.5** The identity law can be expressed with the following strong commutative diagrams:



Obviously,  $\lambda_1 \leq 1$  and  $\lambda_2 \leq 1$ .

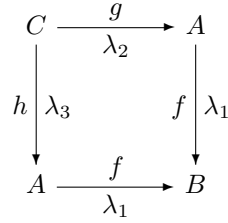
**Isomorphisms** In ordinary mathematics two entities of the same kind can be isomorphic or not. In the fuzzy setting, they can be isomorphic up to some degree and the following definition follows this principle:

**Definition 2.4** Two objects  $A$  and  $B$  are isomorphic to some degree  $\lambda$  if there are arrows  $f : A \xrightarrow{\lambda_1} B$  and  $g : B \xrightarrow{\lambda_2} A$  such that the following diagrams

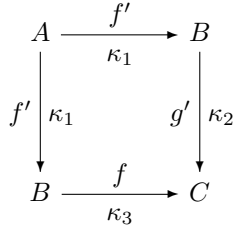


are commutative with degree that is equal to  $\lambda$ , where  $\lambda = \min\{\lambda_1, \lambda_2\}$ .

Arrows can be *monic* or *epic* up to some degree. In particular, an arrow  $f : A \xrightarrow{\lambda_1} B$  is monic up to  $\nu$  if there are two arrows  $g : C \xrightarrow{\lambda_2} A$   $h : C \xrightarrow{\lambda_3} A$  such that the following diagram



commutes with degree equal to  $\nu = \min\{\lambda_1, \lambda_2, \lambda_3\}$ . In addition,  $g =_\nu h$ , that is,  $g$  and  $h$  are equal with degree  $\nu$ . Similarly, an arrow  $f' : A \xrightarrow{\kappa_1} B$  is epic up to  $\nu'$  if there are two arrows  $g' : B \xrightarrow{\kappa_2} C$   $h : B \xrightarrow{\kappa_3} C$  such that the following diagram



commutes with degree equal to  $\nu' = \min\{\kappa_1, \kappa_2, \kappa_3\}$ . In addition,  $g' =_{\nu'} h'$ , that is,  $g'$  and  $h'$  are equal with degree  $\nu'$ .

**Initial and Terminal Objects** An object  $T$  of a fuzzy category  $\mathcal{C}$  is called terminal if there is exactly one arrow  $A \xrightarrow{1} T$  for each object  $A$  of  $\mathcal{C}$ . An object of a fuzzy category that has a unique arrow with plausibility degree equal to one to each object (including itself), is called an *initial object*.

### 3 Conclusions

I have introduced fuzzy categories and some fuzzy categorical structures. There is much work ahead! First one needs to define fuzzy functors, then fuzzy natural transformations. However, the most important of all is to see whether these categories have interesting properties and whether they can be used to solve interesting problems.

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